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On Efficiency Balanced and Variance Balanced Design

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Summary

In this paper, some more results on Efficiency Balanced (EB) and Variance Balanced (VB) designs are obtained. It is shown that a number of EB designs with more than v+1, treatments can be obtained by proper choice of the parameters of the BIBD and the reinforcement parameters. Similarly, a number of VB designs can be constructed by taking different values of n, p, q, x satisfying certain conditions.

Key Words : Efficiency balanced design. Variance balanced design, Balanced incomplete block design, Reinforced design.

Introduction

The efficiency balanced (EB) designs were introduced by Calinski [1] (see also Puri and Nigam [6], and Williams [7]). Let d be an incomplete block design with treatment replication numbers $r_1, r_2, \ldots r_v$ and block sizes $k_1, k_2 \ldots k_b$. Let N (n_{ij}) be the incidence matrix of the design d, and let $r' = (r_1, r_2, \ldots, r_v)$, $k' = (k_1, k_2 \ldots, k_v)$, $R = \text{diag}\,(r_1, r_2 \ldots, r_v)$, $K = \text{diag}\,(k_1, k_2 \ldots, k_b)$, and $C = R - NK^{-1} N'$. The design d is called Generalised Efficiency Balanced (GEB), if for some positive numbers a, $s_1, s_2, \ldots s_v$, the matrix C is of the form

$$C = a\,(S - g^{-1}\,\underline{s}\,\underline{s}') \tag{1.1}$$

where $g = \sum_{i=1}^{v} s_i$, $S = \text{diag}\,(s_1, s_2, \ldots, s_v)$, and $\underline{s}' = (s_1, s_2, \ldots, s_v)$.

From (1.1), it follows that d is GEB design if and only if

$$\sum_{i=1}^{b} n_{ij} \cdot \frac{n_{mj}}{k_j} = c \,.\, s_i \,.\, s_m, \tag{1.2}$$

$$(i \neq m = 1, 2, \ldots, v)$$

where c = a/g. The efficiency E of the design d relative to completely randomised design (CRD) with replication numbers $s_1, s_2, \ldots, s_v$ is given by

$$E = a = c \cdot g = c \cdot \sum_{i=1}^{v} s_i \tag{1.3}$$

If all $s_i = r_i$, $i = 1, 2, \ldots, v$, then the design d is efficiency balanced design and if $s_i = 1$ for all i, then the design d is variance balanced (VB) design.

In this paper, some results on the construction of EB and VB designs through the technique of reinforcement of BIB design are given. In particular, EB designs with more than v + 1 treatments are obtained, where v is the number of treatments of a BIBD which is reinforced.

2. *EB Designs with v + 1 treatments*

Das [2] defined a series of design called reinforced designs. Such designs are obtained by including (i) several additional treatments in each block of existent incomplete block and (ii) any number n (≥ 0) additional blocks each containing all treatments. In Theorem 2.1 an EB design with v + 1 treatments is obtained by reinforcement of BIBD with v treatments.

Theorem 2.1 : Let a BIBD (v, b, r, k, λ) be reinforced by (i) including n (>0) additional blocks each containing each of the v treatments q times and one extra treatment x times and (ii) including the extra treatment p times in each of b blocks. Then the new design is EB if and only if n, p, q, x satisfy

$$\frac{\lambda (vq + x) + nq^2 (k + p)}{rp (vq + x) + nqx (k + p)} = \frac{r + nq}{bp + nx}$$

Proof : The treatments of BIBD will be denoted by i = 1, 2, . . ., v, while the extra treatment by i= v+1. The blocks of the BIBD will be denoted by j = 1, 2, . . ., b while the additional n blocks by j = b+ 1, b+ 2, . . ., b+ n. Clearly, we have

$$r_i = r + nq, \quad i = 1, 2, \ldots, v$$

$$= bp + nx \quad i = v + 1$$

$$k_j = k + p, \quad j = 1, 2, \ldots, b$$

Table 1. EB Designs

Sr. No.	BIBD	Reinforcement parameters	EB Design
1.	v, b, r, k, λ	$n, p = 0, q = 1$ $x = (r-\lambda)/\lambda$	$r'_1 = r + n$, $r'_2 = n(r-\lambda)/\lambda$ $k'_1 = k$, $k'_2 = rk/\lambda$ $E = \frac{\lambda v}{rk} + \frac{n(r-\lambda)}{rk(r+n)}$ (Das and Ghosh [3])
2.	v, b, r, k, λ	$n, p = 0, q = \lambda$ $x = (r-\lambda)$	$r'_1 = r + n\lambda$, $r'_2 = n(r-\lambda)$ $k'_1 = k$, $k'_2 = rk$ $E = \frac{\lambda(v+nk)}{k(r+n\lambda)}$
3.	$v = b = 3$ $r = k = 2, \lambda = 1$	$n = 2, p = 0, x = q$	$r'_1 = r + 2q$, $r'_2 = 2q$ $k'_1 = k$, $k'_2 = 4q$ $E = \frac{3}{4} + \frac{q}{4(q+1)}$ (Williams [7])
4.	$v = s^2 + s + 1$ $b = s(s^2+s+1)$ $r = s(s+1)$ s = prime or prime power	$n, p = 0$ $x = qs$	$r'_1 = s^2 + s + nq$, $r'_2 = nqs$ $k'_1 = s+1$, $k'_2 = q(s+1)^2$ $E = \frac{vr'_1 + nqs}{(s+1)^2(s^2+s+nq)}$
5.	$v = b = 4t+3$ $r = k = 2(t+1)$ $\lambda = t+1$, $4t+3$ = prime or prime power	$n, p = 0,$ $x = q$	$r'_1 = r + nq$, $r'_2 = nq$ $k'_1 = k$, $k'_2 = 2kq$ $E = \frac{vr' + nq}{2kr'_1}$
6.	v, b, r, k, λ	$npq = r - \lambda,$ $x = 0$	$r'_1 = r + nq$, $r'_2 = bq$ $k'_1 = k + p$, $k'_2 = vq$ $E = 1 - \frac{p(r-\lambda)}{(k+p)(r+rp-\lambda)}$ (Result 3.3 of Das and Ghosh [3])

values of n, p, q, x satisfying the condition of Theorem 3.1. Some of these designs are tabulated in Table 2.

Table 2. VB Designs

Sr. No.	BIBD	Reinforcement parameters	VB Design
1	2	3	4
1.	v, b, r, k, λ	$n, p = 0, q = 1$ $x = 1 + \frac{\lambda(v+1)}{(nk-\lambda)}$	$r'_1 = r + n, r'_2 = nx$ $k'_1 = k, k'_2 = vq + x$ $E = [nx(v+1)] / (v + x)$ [Result 5.1 of Das and Ghosh [3]]
2.	$v = b, r = k = 1$ $\lambda = 0$	$n, p = 0, q = x$	$r'_1 = 1 + nq, r'_2 = nq$ $k'_1 = k, k'_2 = q(v+1), E = nq$ [Result 3.1. of Das and Ghosh [3]]
3.	$v = b = 4t + 3,$ $r = k = 2t + 1$ $\lambda = t$ $4t + 3$ = prime or prime power.	$n = q = 1, p = 0$ $x = 4t + 1$	$r'_1 = 2(t+1), r'_2 = 4t + 1$ $k'_1 = 2t + 1 ; k'_2 = t(2t+1)$ $E = \frac{(4t+1)(t+1)}{2(t+1)}$
4.	$v = b = 4t + 3,$ $r = k + 2t + 1$ $\lambda = t.$ $4t + 3$ = prime or Prime power	$n = q = 1, p = 0$ $x = 4t + 5$	$r'_1 = 2(t+1), r'_2 = 4t + 5$ $k'_1 = 2(t+1), k'_2 = 8(t+1)$ $E = \frac{4t+5}{2(t+1)}$
5.	$v = s^2 + s + 1,$ $b = s(s^2 + s + 1)$ $r = s(s+1)$ $k = s + 1, \lambda = s$ s = prime or prime power.	$n = q = 1, p = 0,$ $x = s^3 + (s+1)^2$	$r'_1 = s^2 + s + 1, r'_2 = s^3 + (s+1)^2$ $k'_1 = s + 1, k'_2 = (s+1)(s^2 + s + 2)$ $E = \frac{s^3 + (s+1)^2}{(s+1)(s^2 + s + 2)}$
6.	$v = b = k+1$ $r = k, \lambda = k - 1$	$n = q = 1, p = 0,$ $x = k^2 + k - 1$	$r'_1 = k + 1, r'_2 = k^2 + k - 1$ $k'_1 = k, k'_2 = k(k+2)$ $E = \frac{k^2 + k - 1}{k}$

$$= bp + nx, \qquad i = v+1, v+2, \ldots, v+t.$$

$$k_i = k + tp, \qquad j = 1, 2, \ldots, b.$$

$$= v_q + t_x, \qquad j = b+1, b+2, \ldots, b+n$$

Then, from the elements of the C matrix of the reinforced design, we find that

$$C_{im} = \frac{\lambda}{k+tp} + \frac{nq^2}{vq+tx}, \quad i \neq m = 1, 2, \ldots, v \tag{4.1}$$

$$= \frac{rp}{k+tp} + \frac{nqx}{vq+tx}, \quad \begin{array}{l} i = 1, 2, \ldots, v \\ m = v+1, v+2, \ldots, v+t \end{array} \tag{4.2}$$

$$= \frac{bp^2}{k+tp} + \frac{nx^2}{vq+tx}, \quad i \neq m = v+1, v+2, \ldots, v+t \tag{4.3}$$

The new design is EB if and only if $C_{im} = cr_i r_m$. Hence, the new design is EB if and only if

$$\frac{\lambda}{k+tp} + \frac{nq^2}{vq+tx} = c\,(r+nq)^2 \tag{4.4}$$

$$\frac{rp}{k+tp} + \frac{nqx}{vq+tx} = c\,(r+nq)\,(bp+nx) \tag{4.5}$$

$$\frac{bp^2}{k+tp} + \frac{nx^2}{vq+tx} = c\,(bp+nx)^2 \tag{4.6}$$

Elimination of c from (4.4), (4.5) and (4.6) establishes the theorem.

The efficiency E of this design is given by

$$E = c \sum_{i=1}^{v+t} r_i = c\{v(r+nq) + t(bp+nq)\} \tag{4.7}$$

A number of EB designs with more than (v + 1) treatments can be obtained by proper choice of the parameters of the BIBD and the reinforcement parameters.